

DOCTORAL THESIS

# Towards a Unified Proof of the Foundations of Mathematics

A Categorical and Universal-Algebraic Synthesis

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# Abstract

This thesis proposes a unified foundation for mathematics resting on three pillars: *universal algebra*, which governs operations and the equations they satisfy; *Martin-Löf’s intuitionistic theory of types*, which governs construction and constructive proof; and *topos theory*, which governs the universes in which mathematical discourse is interpreted. The central claim, called here the *Unified Foundations Thesis*, is that these three perspectives are not rival foundations but three presentations of a single categorical structure, and that the passage between them is effected by theorems of an entirely classical character: the functorial semantics of Lawvere, the propositions-as-types correspondence read through the Brouwer–Heyting–Kolmogorov interpretation, the interpretation of dependent type theory in locally cartesian closed categories due to Seely, and the equivalence between intuitionistic higher-order logic and elementary toposes established by Lambek and Scott.

The body of the thesis develops each pillar with full definitions and proofs: Birkhoff’s completeness theorem for equational logic and the variety theorem; the syntax and BHK semantics of dependent type theory with identity types; the internal language of an elementary topos with its Kripke–Joyal semantics; and the three correspondence theorems that close the triangle. A short chapter treats the Kolmogorov–Arnold representation theorem as a result about the generation of operations by superposition, connecting the analytic tradition to the algebraic notion of clone. The synthesis is packaged as a single framework, *Universal Categorical Algebraic Topology* (UCAT), whose scope and limitations — in particular its *relative*, rather than absolute, character in the light of Gödel’s incompleteness theorems — are assessed in the final chapter.

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# Chapter 1

## Introduction

### 1.1 The problem of foundations

The foundational debates of the early twentieth century bequeathed to mathematics not one foundation but several. Set theory, in its Zermelo–Fraenkel formulation, offered a single universe of discourse in which all mathematical objects could be encoded, but at the price of encoding: the set-theoretic definition of an ordered pair or a function tells us nothing about pairs or functions that we use in practice, and everything about a particular simulation of them. Intuitionism, in the tradition of Brouwer and Heyting, offered an account of mathematical activity as *construction*, but for decades lacked a mathematical semantics of comparable generality. Algebra, after Noether, increasingly organized itself around structures and the homomorphisms between them — a tendency made explicit by Birkhoff’s universal algebra [2] and, a decade later, by the category theory of Eilenberg and Mac Lane [6].

By the middle of the century these three currents — the set-theoretic, the constructive, and the structural — had each achieved technical maturity, and each could with some justice claim to be foundational. The present thesis takes the position that the choice between them is a false one. Its guiding claim is the following.

**Unified Foundations Thesis.** Universal algebra, intuitionistic type theory, and topos theory are three presentations of a single underlying structure. Universal algebra presents it *equationally*, type theory presents it *constructively*, and topos theory presents it *semantically*; and the translations between the three presentations are theorems, not analogies.

The thesis is not offered as a slogan. Each of the words “presentation” and “translation” will be given a precise mathematical content, and the translations will be exhibited as explicit theorems: the equivalence between equational theories and Lawvere’s algebraic categories (Chapter 3), the Curry–Howard reading of the Brouwer–Heyting–Kolmogorov interpretation (Chapter 4), the interpretation of dependent type theory in locally cartesian closed categories (Chapter 6), and the Lambek–Scott equivalence between intuitionistic higher-order theories and elementary toposes (Chapter 6).

### 1.2 The three pillars

**Universal algebra and categorical logic.** Universal algebra abstracts from particular algebraic structures — groups, rings, lattices — to the general notion of a *signature* of

operations and a set of *equations* those operations satisfy. Its fundamental results, due chiefly to Birkhoff [2], are of two kinds: a *proof-theoretic* completeness theorem, matching equational derivability with semantic validity, and a *model-theoretic* characterization of the classes of algebras so axiomatized (the HSP theorem). Lawvere’s thesis [16] recast the entire subject functorially: an algebraic theory is a category with finite products, a model is a product-preserving functor, and a homomorphism is a natural transformation. This recasting is the first of our translations, and its details occupy Chapter 3.

**Intuitionistic type theory.** Martin-Löf’s theory of dependent types [22, 23] provides the constructive pillar. Its judgments classify constructions rather than truths; its propositions are types, and its proofs are terms. The Brouwer–Heyting–Kolmogorov interpretation [8, 27], which explains the intuitionistic connectives in terms of what counts as a proof of them, is realized *literally* by the type-theoretic calculus through the Curry–Howard correspondence [5, 11]. Of particular importance for us is Martin-Löf’s treatment of *equality* as a type former, the identity type  $\text{Id}_A(a, b)$ , which internalizes the relation of provable equality and makes equational reasoning itself constructive. Chapter 4 develops the calculus and proves its basic laws.

**Topos theory.** An elementary topos, in the sense of Lawvere and Tierney, is a category that behaves like the category of sets to precisely the extent needed to interpret higher-order intuitionistic logic: it has finite limits, exponentials, and a subobject classifier. Toposes arise as categories of sheaves in geometry [21], as realizability universes in the semantics of computation, and as free constructions from syntax; each such universe carries an *internal language* in which ordinary mathematical arguments can be carried out, subject only to the discipline of intuitionistic reasoning [12, 13, 15]. Chapter 5 develops the internal language and its Kripke–Joyal semantics in full.

### 1.3 The two instruments

Two further tools recur throughout, and the original 2004 text singled them out by name.

The first is the *Brouwer–Heyting–Kolmogorov interpretation* itself, which serves in this thesis as the semantic bridge between the type-theoretic and topos-theoretic pillars: the BHK clauses, read inside a topos by the Kripke–Joyal semantics, are theorems; read as a calculus of constructions, they are the typing rules of Chapter 4.

The second is the *Kolmogorov–Arnold representation theorem* [14, 1], which asserts that every continuous function of  $n$  real variables is a superposition of continuous functions of one variable and addition. Within the architecture of this thesis the theorem plays a conceptual role: it is the analytic counterpart of the universal-algebraic idea that all operations of a structure are generated by composition from a small stock of basic ones — the idea made precise by the notion of a *clone*. Chapter 7 makes this reading exact.

### 1.4 Outline and conventions

Chapter 2 fixes categorical language: categories, functors, natural transformations, the Yoneda lemma, adjunctions, limits, and cartesian closed categories. Chapter 3 develops universal algebra to the point of Birkhoff’s two theorems and their functorial reformulation. Chapter 4 develops Martin-Löf type theory and the BHK interpretation. Chapter 5 develops elementary toposes, the Mitchell–Bénabou language, and Kripke–Joyal semantics.

Chapter 6 proves the three correspondence theorems that close the triangle. Chapter 7 treats the Kolmogorov–Arnold theorem. Chapter 8 assembles the Unified Foundations Theorem, names the resulting framework UCAT, and discusses its scope and limitations.

Throughout, “logic” means intuitionistic logic unless classical logic is expressly invoked; **Set** denotes the category of sets and functions; and composition is written in the order  $g \circ f$ . All categories are locally small unless stated otherwise.



## Chapter 2

# Categorical Preliminaries

This chapter collects the categorical background used throughout. Proofs are given where they are short and instructive; for the remainder we cite Mac Lane [20].

### 2.1 Categories, functors, natural transformations

**Definition 2.1.** A category  $\mathbf{C}$  consists of a collection of objects, for each pair of objects  $A, B$  a set  $\text{Hom}_{\mathbf{C}}(A, B)$  of morphisms, an associative composition  $\circ: \text{Hom}(B, C) \times \text{Hom}(A, B) \rightarrow \text{Hom}(A, C)$ , and for each object  $A$  an identity  $1_A$  neutral for composition.

**Definition 2.2.** A functor  $F: \mathbf{C} \rightarrow \mathbf{D}$  assigns objects to objects and morphisms to morphisms, preserving identities and composition. A natural transformation  $\eta: F \Rightarrow G$  between functors  $F, G: \mathbf{C} \rightarrow \mathbf{D}$  assigns to each object  $A$  a morphism  $\eta_A: FA \rightarrow GA$  such that for every  $f: A \rightarrow B$  the square  $Gf \circ \eta_A = \eta_B \circ Ff$  commutes.

Functors  $\mathbf{C} \rightarrow \mathbf{D}$  and natural transformations between them form the functor category  $[\mathbf{C}, \mathbf{D}]$ . The category  $[\mathbf{C}^{\text{op}}, \mathbf{Set}]$  of *presheaves* on  $\mathbf{C}$  plays a distinguished role, mediated by the Yoneda embedding  $y: \mathbf{C} \rightarrow [\mathbf{C}^{\text{op}}, \mathbf{Set}]$ ,  $yA = \text{Hom}_{\mathbf{C}}(-, A)$ .

**Lemma 2.3** (Yoneda). *For every presheaf  $P: \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$  and every object  $A$  of  $\mathbf{C}$  there is a bijection*

$$\text{Nat}(yA, P) \cong P(A),$$

*natural in both  $A$  and  $P$ .*

*Proof.* Given  $\eta: yA \Rightarrow P$ , set  $\Phi(\eta) = \eta_A(1_A) \in P(A)$ . Conversely, given  $x \in P(A)$ , define  $\Psi(x)_B: \text{Hom}(B, A) \rightarrow P(B)$  by  $\Psi(x)_B(f) = P(f)(x)$ ; naturality of  $\Psi(x)$  is functoriality of  $P$ . For one composite,  $\Phi(\Psi(x)) = \Psi(x)_A(1_A) = P(1_A)(x) = x$ . For the other, let  $\eta: yA \Rightarrow P$  and  $f: B \rightarrow A$ ; naturality of  $\eta$  at  $f$  gives  $\eta_B(1_A \circ f) = P(f)(\eta_A(1_A))$ , i.e.  $\eta_B(f) = \Psi(\Phi(\eta))_B(f)$ . Naturality in  $A$  and  $P$  is a routine diagram chase.  $\square$

**Corollary 2.4.** *The Yoneda embedding is full and faithful. In particular, two objects with naturally isomorphic hom-functors are isomorphic: an object is determined, up to isomorphism, by the totality of its relationships to all objects.*

The corollary is the structuralist credo in mathematical form, and it will return in the final chapter: it is the precise sense in which the *relational* description of an object is complete.

## 2.2 Limits, adjunctions, exponentials

**Definition 2.5.** A limit of a diagram  $D: \mathbf{J} \rightarrow \mathbf{C}$  is a terminal object in the category of cones over  $D$ . Finite products, equalizers, and pullbacks are the limits over the evident finite shapes; a category with a terminal object and pullbacks has all finite limits.

**Definition 2.6.** An adjunction  $F \dashv G$  between functors  $F: \mathbf{C} \rightarrow \mathbf{D}$  and  $G: \mathbf{D} \rightarrow \mathbf{C}$  is a bijection  $\text{Hom}_{\mathbf{D}}(FA, B) \cong \text{Hom}_{\mathbf{C}}(A, GB)$  natural in  $A$  and  $B$ .

**Definition 2.7.** A category  $\mathbf{C}$  with finite products is cartesian closed if for every object  $B$  the functor  $- \times B$  has a right adjoint  $(-)^B$ ; the counit  $\text{ev}: A^B \times B \rightarrow A$  is evaluation. It is locally cartesian closed if every slice  $\mathbf{C}/A$  is cartesian closed.

*Example 2.8.* **Set** is locally cartesian closed:  $A^B$  is the set of functions from  $B$  to  $A$ , and the slice **Set**/ $I$  is equivalent to the category of  $I$ -indexed families of sets, with exponentials computed fibrewise.

Cartesian closure is the categorical shadow of implication, and local cartesian closure of universal quantification; this remark will be made a theorem in Chapters 5 and 6. Right adjoints preserve limits and left adjoints colimits [20]; we use this freely.

## 2.3 Equivalence of categories

The correspondence theorems of Chapter 6 assert *equivalences*, and the notion deserves a moment's care.

**Definition 2.9.** A functor  $F: \mathbf{C} \rightarrow \mathbf{D}$  is an equivalence if there is a functor  $G: \mathbf{D} \rightarrow \mathbf{C}$  with natural isomorphisms  $GF \cong 1_{\mathbf{C}}$  and  $FG \cong 1_{\mathbf{D}}$ . Equivalently [20]:  $F$  is full, faithful, and essentially surjective (every object of  $\mathbf{D}$  is isomorphic to some  $FA$ ).

Equivalence, not isomorphism, is the right notion of “sameness” for categories, for the same reason that isomorphism, not equality, is the right notion for objects: it demands agreement of all structure that is invariant under the maps the theory itself provides, and nothing more. A category is equivalent to any of its *skeletons* (one object per isomorphism class), so equivalence deliberately ignores how many isomorphic copies of an object are present — information no mathematical argument ever uses. When Theorem 3.7 asserts  $\text{Mod}(\mathcal{T}) \simeq \text{Mod}(\mathbf{L}_{\mathcal{T}})$ , it asserts that every statement about models, homomorphisms, and their composites transfers across the translation intact; it does not assert a bijection of underlying classes, which would be both false and beside the point. This calibration of “the same” — strict at the level of morphisms, relaxed at the level of objects — is itself part of the foundational proposal of this thesis.

## 2.4 Adjunctions as the grammar of foundations

Three families of adjunctions organize everything that follows, and it is worth displaying them together before the detailed work begins; the observation that logical and mathematical structure is systematically given by adjoint functors is Lawvere’s [18].

*Example 2.10* (Free constructions). The forgetful functor from groups (or monoids, rings, modules, or the models of any algebraic theory) to **Set** has a left adjoint, the free construction. Chapter 3 proves this in the general case (Proposition 3.9) and shows that the free model on the generators is at once a syntactic object — terms modulo provable equality — and a semantic one, initial among models. The unit of the adjunction is the insertion of generators; the counit evaluates formal expressions.

*Example 2.11* (Propositional connectives). In a Heyting algebra — the propositional face of intuitionism, and the structure carried by the subobjects of any object in a topos (Proposition 5.3) — conjunction and implication are related by the adjunction

$$c \leq a \Rightarrow b \quad \text{iff} \quad c \wedge a \leq b,$$

which is cartesian closure specialized to posets. Disjunction is left adjoint to the diagonal, conjunction right adjoint to it. The deduction theorem of logic is thus literally an adjunction.

*Example 2.12* (Quantifiers). For a function  $f: B \rightarrow A$ , precomposition  $f^*: \mathcal{P}(A) \rightarrow \mathcal{P}(B)$  (read: substitution of  $f$  in a predicate) has both adjoints:

$$\exists_f \dashv f^* \dashv \forall_f, \quad \exists_f S = \{a \mid \exists b. fb = a \wedge b \in S\}, \quad \forall_f S = \{a \mid \forall b. fb = a \Rightarrow b \in S\}.$$

Taking  $f$  to be a product projection recovers the ordinary quantifiers. That quantification is adjoint to substitution is the single most useful sentence in categorical logic; Theorems 5.4 and 6.1 are its topos-theoretic and type-theoretic elaborations.

## Chapter 3

# Universal Algebra and Functorial Semantics

### 3.1 Signatures, terms, equations

**Definition 3.1.** A signature  $\Sigma$  is a set of operation symbols each with a finite arity. Given a countable set  $X$  of variables, the set  $T_\Sigma(X)$  of terms is generated inductively: every variable is a term, and if  $\sigma \in \Sigma$  has arity  $n$  and  $t_1, \dots, t_n$  are terms then  $\sigma(t_1, \dots, t_n)$  is a term. An equation is a pair of terms, written  $t \approx s$ . An algebraic theory (in the classical sense) is a pair  $\mathcal{T} = (\Sigma, E)$  of a signature and a set of equations.

**Definition 3.2.** A  $\Sigma$ -algebra  $A$  is a set  $|A|$  together with, for each  $n$ -ary  $\sigma \in \Sigma$ , a function  $\sigma^A: |A|^n \rightarrow |A|$ . Terms are interpreted by structural recursion: an assignment  $v: X \rightarrow |A|$  extends uniquely to  $\hat{v}: T_\Sigma(X) \rightarrow |A|$ . The algebra  $A$  satisfies  $t \approx s$ , written  $A \models t \approx s$ , if  $\hat{v}(t) = \hat{v}(s)$  for every assignment  $v$ . A model of  $\mathcal{T} = (\Sigma, E)$  is a  $\Sigma$ -algebra satisfying every equation of  $E$ ; a homomorphism is a function commuting with all operations. Models of  $\mathcal{T}$  and their homomorphisms form the category  $\text{Mod}(\mathcal{T})$ .

*Example 3.3.* Groups:  $\Sigma = \{\cdot(2), {}^{-1}(1), e(0)\}$  with the usual three equations.  $\text{Mod}(\mathcal{T})$  is the category of groups. Similarly for monoids, rings, lattices, Boolean algebras, and modules over a fixed ring.

### 3.2 Equational logic and Birkhoff's completeness theorem

Equational logic has five rules: reflexivity  $t \approx t$ ; symmetry and transitivity of  $\approx$ ; *congruence*, from  $t_1 \approx s_1, \dots, t_n \approx s_n$  infer  $\sigma(t_1, \dots, t_n) \approx \sigma(s_1, \dots, s_n)$ ; and *substitution*, from  $t \approx s$  infer  $t[u/x] \approx s[u/x]$  for any term  $u$ . Write  $E \vdash t \approx s$  for derivability from the axioms  $E$ .

**Theorem 3.4** (Birkhoff completeness, 1935).  $E \vdash t \approx s$  if and only if every model of  $E$  satisfies  $t \approx s$ .

*Proof.* *Soundness* is induction on derivations: each rule preserves satisfaction in an arbitrary model; for substitution one uses the composition law  $\hat{v}(t[u/x]) = v[x \mapsto \hat{v}(u)](t)$ , itself proved by structural induction on  $t$ .

*Completeness.* Define on  $T_\Sigma(X)$  the relation  $t \sim s \iff E \vdash t \approx s$ . The reflexivity, symmetry, and transitivity rules make  $\sim$  an equivalence relation, and the congruence rule

makes it a *congruence*: compatible with every operation. The quotient  $F = T_\Sigma(X)/\sim$  therefore carries a  $\Sigma$ -algebra structure with  $\sigma^F([t_1], \dots, [t_n]) = [\sigma(t_1, \dots, t_n)]$ .

$F$  is a model of  $E$ : given an equation  $(t \approx s) \in E$  and any assignment  $w: X \rightarrow F$ , choose representatives  $w(x) = [u_x]$ ; then  $\hat{w}(t) = [t[u_x/x]_x]$  by induction on  $t$ , and the substitution rule (applied once per variable) yields  $E \vdash t[u_x/x]_x \approx s[u_x/x]_x$ , whence  $\hat{w}(t) = \hat{w}(s)$ .

Now suppose every model satisfies  $t \approx s$ . Apply this to  $F$  with the tautological assignment  $v_0(x) = [x]$ : then  $[t] = \hat{v}_0(t) = \hat{v}_0(s) = [s]$ , which is precisely  $E \vdash t \approx s$ .  $\square$

The algebra  $F$  constructed above is the *free* model on  $X$ ; the proof exhibits freeness as the engine of completeness. The same circle of ideas yields the model-theoretic half of Birkhoff's legacy:

**Theorem 3.5** (HSP theorem [2]). *A class of  $\Sigma$ -algebras is definable by equations if and only if it is closed under homomorphic images, subalgebras, and products.*

We use the HSP theorem only for orientation and omit the proof (see [3]).

### 3.3 Lawvere theories and functorial semantics

Lawvere's 1963 thesis [16] replaces the syntactic pair  $(\Sigma, E)$  by a category.

**Definition 3.6.** *A Lawvere theory is a category  $\mathbf{L}$  whose objects are the natural numbers, in which  $n$  is the  $n$ -fold product of 1. A model of  $\mathbf{L}$  in a category  $\mathbf{C}$  with finite products is a finite-product-preserving functor  $M: \mathbf{L} \rightarrow \mathbf{C}$ ; a homomorphism of models is a natural transformation. Write  $\text{Mod}(\mathbf{L}, \mathbf{C})$  for the resulting category, and  $\text{Mod}(\mathbf{L})$  for  $\text{Mod}(\mathbf{L}, \mathbf{Set})$ .*

The morphisms  $n \rightarrow 1$  of  $\mathbf{L}$  are the abstract  $n$ -ary operations of the theory; the equations have been absorbed into the composition of the category, so that two terms denote the same morphism exactly when they are provably equal. This is the sense — made precise in the next theorem, whose proof we give in outline — in which “an algebraic equation is a commutative diagram.”

**Theorem 3.7** (Functorial semantics). *For every classical algebraic theory  $\mathcal{T} = (\Sigma, E)$  there is a Lawvere theory  $\mathbf{L}_{\mathcal{T}}$  and an equivalence of categories*

$$\text{Mod}(\mathcal{T}) \simeq \text{Mod}(\mathbf{L}_{\mathcal{T}}).$$

*Conversely every Lawvere theory arises in this way from some  $(\Sigma, E)$ .*

*Proof sketch.* Define  $\text{Hom}_{\mathbf{L}_{\mathcal{T}}}(n, m)$  to be the set of  $m$ -tuples of  $\sim$ -classes of terms in variables  $x_1, \dots, x_n$ , where  $\sim$  is provable equality as in Theorem 3.4; composition is substitution, which is well defined on classes by the substitution rule, and associativity of composition is associativity of substitution. Products are given by concatenation of variable lists.

Given a model  $A$  of  $\mathcal{T}$ , define  $M_A: \mathbf{L}_{\mathcal{T}} \rightarrow \mathbf{Set}$  on objects by  $M_A(n) = |A|^n$  and on a class  $[t]: n \rightarrow 1$  by the interpretation  $t^A: |A|^n \rightarrow |A|$ ; this is well defined by soundness and functorial by the substitution law. Conversely a product-preserving  $M$  yields the algebra  $|A| = M(1)$  with  $\sigma^A = M([\sigma(x_1, \dots, x_n)])$ , a model of  $E$  by completeness. The two constructions are mutually inverse up to natural isomorphism, and homomorphisms correspond to natural transformations by naturality at the object 1. For the converse statement, take as  $\Sigma$  all morphisms  $n \rightarrow 1$  of  $\mathbf{L}$  and as  $E$  all equations valid in  $\mathbf{L}$ .  $\square$

*Remark 3.8.* Because a model is simply a product-preserving functor, the definition makes sense with any finite-product category  $\mathbf{C}$  as codomain: groups in topological spaces are topological groups, groups in a topos are internal groups. Algebra has become *portable across universes*; this is the hinge on which Chapter 8 turns.

**Proposition 3.9.** *The forgetful functor  $U: \text{Mod}(\mathbf{L}) \rightarrow \mathbf{Set}$ ,  $M \mapsto M(1)$ , has a left adjoint, and  $\text{Mod}(\mathbf{L})$  has all limits and colimits.*

*Proof sketch.* The free model on a set  $X$  is the filtered colimit over finite  $X_0 \subseteq X$  of the representable-built models  $\text{Hom}_{\mathbf{L}}(|X_0|, -)$ ; alternatively, the term-algebra construction of Theorem 3.4 transported along Theorem 3.7. Limits are computed pointwise; colimits exist by the adjoint functor theorem,  $\text{Mod}(\mathbf{L})$  being complete, well-powered, and with a cogenerating set [20].  $\square$

### 3.4 Morphisms of theories

The functorial formulation earns its keep when theories themselves are allowed to vary.

**Definition 3.10.** *A morphism of Lawvere theories  $\Phi: \mathbf{L} \rightarrow \mathbf{L}'$  is a functor which is the identity on objects and preserves finite products.*

Precomposition with  $\Phi$  restricts models:  $\Phi^*: \text{Mod}(\mathbf{L}') \rightarrow \text{Mod}(\mathbf{L})$ . Every “forgetful” functor of algebra that forgets *structure* rather than *properties* arises this way: rings to abelian groups, abelian groups to groups, groups to monoids, monoids to sets (via the inclusion of the trivial theory).

**Proposition 3.11.** *For any morphism of theories  $\Phi$ , the restriction functor  $\Phi^*$  has a left adjoint  $\Phi_!$ .*

*Proof sketch.* On the free model on a set  $X$ , let  $\Phi_!$  return the free  $\mathbf{L}'$ -model on  $X$ ; freeness forces the definition on morphisms between frees, and an arbitrary model, being a coequalizer of frees (present it by generators and relations via Proposition 3.9), is sent to the corresponding coequalizer in  $\text{Mod}(\mathbf{L}')$ , which exists by the same proposition. Adjointness is checked on frees and propagates along colimits.  $\square$

*Example 3.12.* For the abelianization morphism (from the theory of groups to the theory of abelian groups, sending the operations to themselves),  $\Phi_!$  is  $G \mapsto G/[G, G]$ ; for monoids  $\rightarrow$  groups,  $\Phi_!$  is the group completion. The classical constructions of algebra are thus not ad hoc: each is *the* left adjoint that the framework predicts, unique up to unique isomorphism once its universal property is stated.

## Chapter 4

# Martin-Löf Type Theory and the BHK Interpretation

### 4.1 The BHK interpretation

Intuitionistic logic, as codified by Heyting [8], explains the meaning of a compound proposition by saying what counts as a *proof* of it. The Brouwer–Heyting–Kolmogorov reading is:

- a proof of  $A \wedge B$  is a pair consisting of a proof of  $A$  and a proof of  $B$ ;
- a proof of  $A \vee B$  is a proof of  $A$  or a proof of  $B$ , together with the information which;
- a proof of  $A \rightarrow B$  is a construction transforming any proof of  $A$  into a proof of  $B$ ;
- nothing is a proof of  $\perp$ ; and  $\neg A$  abbreviates  $A \rightarrow \perp$ ;
- a proof of  $\forall x:S. A(x)$  is a construction assigning to each  $s$  in  $S$  a proof of  $A(s)$ ;
- a proof of  $\exists x:S. A(x)$  is a pair of a witness  $s$  and a proof of  $A(s)$ .

The interpretation is informal — “construction” is undefined — and the achievement of type theory is to make it formal without falsifying it.

### 4.2 The calculus of dependent types

Martin-Löf’s theory [22, 23] is a calculus of *judgments*:  $\Gamma \vdash A$  **type** (in context  $\Gamma$ ,  $A$  is a type) and  $\Gamma \vdash a : A$  (the term  $a$  inhabits  $A$ ), together with judgmental equalities  $\Gamma \vdash A = B$  and  $\Gamma \vdash a = b : A$ . A *context* is a list  $x_1:A_1, \dots, x_n:A_n$  in which each type may depend on the variables before it. The type formers relevant to us:

**Dependent product.** From  $x:A \vdash B(x)$  **type**, form  $\prod_{x:A} B(x)$ , with introduction  $\lambda x. b$  and elimination (application) obeying  $(\lambda x. b) a = b[a/x]$ . When  $B$  does not depend on  $x$  this is the function type  $A \rightarrow B$ .

**Dependent sum.** From the same data, form  $\sum_{x:A} B(x)$ , with pairing  $(a, b)$  and projections. When  $B$  is constant this is  $A \times B$ .

**Finite types.** The empty type  $\mathbf{0}$  (with its vacuous eliminator), the unit type  $\mathbf{1}$ , and binary sums  $A + B$  with case analysis.

**Natural numbers.**  $\mathbb{N}$  with 0, successor, and the recursor; we use it only glancingly.

**Identity types.** For  $a, b : A$ , a type  $\text{Id}_A(a, b)$ ; introduction  $\text{refl}_a : \text{Id}_A(a, a)$ ; elimination ( $J$ ): given  $x, y : A$ ,  $p : \text{Id}_A(x, y) \vdash C(x, y, p)$  type and a term  $d(x) : C(x, x, \text{refl}_x)$ , there is  $J(d; a, b, p) : C(a, b, p)$  with  $J(d; a, a, \text{refl}_a) = d(a)$ .

Under *propositions as types*, a proposition is a type and its proofs are its terms:  $\wedge$  is  $\times$ ,  $\vee$  is  $+$ ,  $\rightarrow$  is  $\rightarrow$ ,  $\perp$  is  $\mathbf{0}$ ,  $\forall$  is  $\prod$ ,  $\exists$  is  $\sum$ . The BHK clauses are then no longer a gloss but the introduction rules themselves [5, 11, 23].

### 4.3 Constructive validities

We record, with explicit proof terms, the laws used later; each proof is a single well-typed term, and this concreteness is the point.

**Lemma 4.1** (Contraposition). *For any types  $A, B$  there is a term of  $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$ .*

*Proof.*  $\lambda f. \lambda g. \lambda a. g (f a)$ . Given  $f : A \rightarrow B$ ,  $g : B \rightarrow \mathbf{0}$ , and  $a : A$ , the term  $g (f a)$  inhabits  $\mathbf{0}$ ; abstracting yields the displayed type. (The converse implication is not derivable, as the topological semantics of Chapter 5 shows; intuitionistic logic is not symmetric under contraposition.)  $\square$

**Lemma 4.2** (Equality is a congruence). *For every  $f : A \rightarrow B$  there is a term  $\text{ap}_f : \prod_{x, y : A} (\text{Id}_A(x, y) \rightarrow \text{Id}_B(fx, fy))$ , and for every dependent type  $P$  over  $A$  a transport term  $\text{subst} : \prod_{x, y : A} (\text{Id}_A(x, y) \rightarrow P(x) \rightarrow P(y))$ . Moreover  $\text{Id}$  is symmetric and transitive.*

*Proof.* All are instances of  $J$ . For  $\text{ap}_f$  take  $C(x, y, p) = \text{Id}_B(fx, fy)$  and  $d(x) = \text{refl}_{fx}$ . For  $\text{subst}$  take  $C(x, y, p) = P(x) \rightarrow P(y)$  and  $d(x) = \lambda u. u$ . Symmetry:  $C(x, y, p) = \text{Id}_A(y, x)$ ,  $d(x) = \text{refl}_x$ ; transitivity similarly.  $\square$

*Remark 4.3.* Lemma 4.2 is the type-theoretic image of the congruence and substitution rules of equational logic (Theorem 3.4): the five rules of Birkhoff's calculus are exactly  $\text{refl}$ , symmetry, transitivity,  $\text{ap}$ , and  $\text{subst}$ . An algebraic theory can accordingly be *axiomatized inside* type theory — a signature becomes a type equipped with operation terms, the equations become inhabitants of identity types — and equational derivations become proof terms. This observation, elementary as it is, is one leg of the triangle assembled in Chapter 8.

### 4.4 Constructive reasoning in practice

Three specimens indicate how the calculus behaves as a logic, and where it parts company with the classical tradition.

**Proposition 4.4** (Double negation of excluded middle). *For every type  $A$  there is a term of  $\neg\neg(A + \neg A)$ , although  $A + \neg A$  itself is not inhabited for general  $A$ .*



*Proof.* Let  $k : \neg(A + \neg A)$ ; we must produce an inhabitant of  $\mathbf{0}$ . First,  $\lambda a. k(\text{inl } a)$  inhabits  $\neg A$ ; call it  $u$ . Then  $k(\text{inr } u)$  inhabits  $\mathbf{0}$ . The full term is  $\lambda k. k(\text{inr } (\lambda a. k(\text{inl } a)))$ . That excluded middle is not derivable outright follows from the topological semantics of Chapter 5 (Example 5.2(iii)): a derivation would force every subobject in every topos to be complemented. Intuitionistic logic thus refutes no classical theorem — it declines to assert some.  $\square$

*Remark 4.5* (The existence property). A closed proof of  $\exists x:S. A(x)$  is, by the BHK clause for  $\Sigma$ , literally a pair: it *contains* a witness and can be evaluated to extract it. Classical set theory has no such property — a classical proof of existence may proceed by contradiction and exhibit nothing. This is not a defect repaired by type theory but a different contract between prover and reader, and it is what makes the calculus of this chapter executable: proofs compute.

*Remark 4.6* (Choice). The type-theoretic statement  $\prod_{x:A} \sum_{y:B} C(x, y) \rightarrow \sum_{f:A \rightarrow B} \prod_{x:A} C(x, fx)$  is provable outright — the “axiom” of choice is a theorem here, because a proof of the hypothesis already carries the choice function. The set-theoretic axiom of choice, by contrast, corresponds to a strong completeness assertion about the semantic universe, and in a topos it implies Booleanness (Diaconescu’s argument; see [13]). The divergence between the two readings is a measure of the distance between proof-relevant and proof-irrelevant mathematics.

*Remark 4.7* (On extensionality). Two regimes for the identity type are available: the *intensional* theory, in which  $\text{Id}$  is governed only by  $J$ , and the *extensional* theory of [23], in which an inhabitant of  $\text{Id}_A(a, b)$  licenses the judgmental equality  $a = b : A$ . The semantics of Chapter 6 interprets the extensional theory; the subtleties of interpreting the intensional theory are noted there (Remark 6.2).

## Chapter 5

# Topos Theory and Internal Logic

### 5.1 Elementary toposes

**Definition 5.1.** An elementary topos is a category  $\mathcal{E}$  with finite limits and exponentials, equipped with a subobject classifier: an object  $\Omega$  and a morphism  $\text{true}: 1 \rightarrow \Omega$  such that for every monomorphism  $m: S \rightarrowtail A$  there is a unique characteristic morphism  $\chi_m: A \rightarrow \Omega$  making

$$\begin{array}{ccc} S & \longrightarrow & 1 \\ \downarrow m & & \downarrow \text{true} \\ A & \xrightarrow{\chi_m} & \Omega \end{array}$$

a pullback.

*Example 5.2.* (i) **Set**, with  $\Omega = \{0, 1\}$ . (ii) Presheaves  $[\mathbf{C}^{\text{op}}, \mathbf{Set}]$ , with  $\Omega(A)$  the set of sieves on  $A$ . (iii) Sheaves on a topological space  $X$ , where  $\Omega$  is the sheaf of open subsets; here the internal logic is genuinely intuitionistic:  $\neg\neg U = \text{int}(\overline{U})$  need not equal  $U$ , so excluded middle fails. See [21, 12].

**Proposition 5.3.** In any topos,  $\text{Sub}(A)$ , the poset of subobjects of  $A$ , is a Heyting algebra, and pullback along any  $f: B \rightarrow A$  is a Heyting homomorphism  $f^*: \text{Sub}(A) \rightarrow \text{Sub}(B)$  possessing both adjoints  $\exists_f \dashv f^* \dashv \forall_f$ . Correspondingly  $\Omega$  is an internal Heyting algebra.

*Proof sketch.* Meets are pullbacks of subobjects; the top element is  $A$  itself; joins and the Heyting implication are constructed from  $\Omega$ 's structure, e.g.  $\Rightarrow: \Omega \times \Omega \rightarrow \Omega$  classifies the equalizer of  $\wedge$  and the first projection. The adjoints to pullback exist by a classical argument using the exponentials and the classifier [21]. Distributivity, hence Heyting-ness, follows from the adjunctions.  $\square$

### 5.2 The Mitchell–Bénabou language

A topos carries an internal language. Its *types* are the objects of  $\mathcal{E}$ ; its terms are built from typed variables by the operations available in  $\mathcal{E}$ : pairing into products, application of morphisms, evaluation of exponentials, and the propositional operations of Proposition 5.3 on terms of type  $\Omega$ . A *formula* with free variables  $x_1:A_1, \dots, x_n:A_n$  is a term  $\varphi$  of type  $\Omega$  in those variables, i.e. a morphism  $A_1 \times \dots \times A_n \rightarrow \Omega$ ; its *extension*  $\{\vec{x} \mid \varphi\}$  is the subobject classified by that morphism. Quantifiers are interpreted by the adjoints of Proposition 5.3:  $\forall x_n.\varphi$  is  $\forall_\pi$  applied to the extension of  $\varphi$ , where  $\pi$  deletes the last variable, and dually for  $\exists$ . A formula is *valid*, written  $\mathcal{E} \models \varphi$ , if its extension is the maximal subobject.

### 5.3 Kripke–Joyal semantics

Validity unwinds into a forcing calculus. For  $U$  an object, a *generalized element* of  $A$  at stage  $U$  is a morphism  $\alpha: U \rightarrow A$ ; write  $U \Vdash \varphi(\alpha)$  when  $\alpha$  factors through the extension of  $\varphi$ .

**Theorem 5.4** (Kripke–Joyal). *For all formulas  $\varphi, \psi$  and  $\alpha: U \rightarrow A$ :*

1.  $U \Vdash (\varphi \wedge \psi)(\alpha)$  iff  $U \Vdash \varphi(\alpha)$  and  $U \Vdash \psi(\alpha)$ ;
2.  $U \Vdash (\varphi \vee \psi)(\alpha)$  iff there is an epimorphic family  $p_i: U_i \rightarrow U$  with, for each  $i$ ,  $U_i \Vdash \varphi(\alpha p_i)$  or  $U_i \Vdash \psi(\alpha p_i)$ ;
3.  $U \Vdash (\varphi \Rightarrow \psi)(\alpha)$  iff for every  $p: V \rightarrow U$ ,  $V \Vdash \varphi(\alpha p)$  implies  $V \Vdash \psi(\alpha p)$ ;
4.  $U \Vdash (\exists y: B. \varphi)(\alpha)$  iff there is an epi  $p: V \twoheadrightarrow U$  and  $\beta: V \rightarrow B$  with  $V \Vdash \varphi(\alpha p, \beta)$ ;
5.  $U \Vdash (\forall y: B. \varphi)(\alpha)$  iff for every  $p: V \rightarrow U$  and every  $\beta: V \rightarrow B$ ,  $V \Vdash \varphi(\alpha p, \beta)$ .

*Proof of clause 3, as a sample.* The extension  $E_{\varphi \Rightarrow \psi}$  is constructed so that a generalized element factors through it iff its pullback of  $E_\varphi$  is contained in its pullback of  $E_\psi$  as subobjects of  $U$  (Proposition 5.3, adjointness of Heyting implication). Containment of subobjects of  $U$  is equivalent to the stated quantification over all  $p: V \rightarrow U$ : one direction is composition; for the other, take  $p$  to be the inclusion of the pullback of  $E_\varphi$  itself. The remaining clauses are analogous, the existential requiring the epi–mono factorization of  $\mathcal{E}$ . Full details in [21].  $\square$

The clauses are precisely the BHK clauses, with “construction” replaced by “morphism” and provability localized to stages: the informal constructive semantics of Chapter 4 has become a theorem about every topos.

**Theorem 5.5** (Soundness of intuitionistic logic). *Every sequent derivable in intuitionistic higher-order logic is valid in every elementary topos. Classical logic is valid iff the topos is Boolean ( $\Omega$  an internal Boolean algebra), which Example 5.2(iii) shows is not automatic.*

*Proof sketch.* Induction on derivations; each rule of the intuitionistic sequent calculus translates to an order-theoretic fact about the Heyting algebras  $\text{Sub}(-)$  and the adjunctions of Proposition 5.3 — e.g. the  $\rightarrow$ -introduction rule is exactly the adjunction defining Heyting implication, and the quantifier rules are the adjunctions  $\exists_f \dashv f^* \dashv \forall_f$  together with the Beck–Chevalley condition ensuring substitution commutes with quantification [15, 13].  $\square$

### 5.4 A worked example: forcing over a space

To see the semantics do something, let  $X$  be a topological space and  $\mathcal{E} = \text{Sh}(X)$  its topos of sheaves. Stages may be taken to be the open sets  $U \subseteq X$  (the subterminal objects), and the clauses of Theorem 5.4 specialize as follows: epimorphic families are open covers, so

$$U \Vdash \varphi \vee \psi \iff U = \bigcup_i U_i \text{ with each } U_i \Vdash \varphi \text{ or } U_i \Vdash \psi,$$

and implication quantifies over open subsets  $V \subseteq U$ . Truth is *local*: a disjunction may hold on  $U$  without either disjunct holding on all of  $U$ .

Concretely, take  $X = \mathbb{R}$  and let  $\varphi$  assert, of a continuous real-valued function  $f$  (a generalized element of the sheaf of continuous functions), that  $f$  is invertible or  $1 - f$  is

invertible. Around each point of  $\mathbb{R}$ , at least one of  $f, 1 - f$  is bounded away from zero on a neighbourhood, so *every*  $U$  is covered by opens on which one disjunct holds:  $U \Vdash \varphi$  always. Yet on no neighbourhood of a point where  $f = 0$  is the first disjunct true, and dually; globally, neither disjunct need hold. Excluded middle fails for the same reason:  $U \Vdash \neg\neg\sigma$  says  $\sigma$  holds on a dense collection of opens, while  $U \Vdash \sigma \vee \neg\sigma$  demands a cover deciding  $\sigma$  on each piece — and  $\neg\neg U = \text{int}(\overline{U})$  records exactly the gap.

When  $X$  is replaced by a poset  $P$  with the Alexandrov topology, the clauses collapse to Kripke’s relational semantics for intuitionistic logic; when  $P$  is a forcing partial order, they recover, on the double-negation sheaves, Cohen’s forcing. The single calculus of Theorem 5.4 thus subsumes the topological, relational, and set-theoretic semantics of intuitionism at one stroke [21].

## Chapter 6

# The Correspondence Theorems

The three pillars are now erected; this chapter builds the passages between them. Together with Theorem 3.7 (algebra  $\leftrightarrow$  categories with finite products) they close a triangle whose vertices are the equational, constructive, and semantic presentations.

### 6.1 Type theory in locally cartesian closed categories

The interpretation, due to Seely [26] with antecedents in Lawvere’s hyperdoctrines [18, 19], reads a context  $\Gamma$  as an object  $\llbracket \Gamma \rrbracket$ , a dependent type  $\Gamma \vdash A$  as an object of the slice  $\mathcal{E}/\llbracket \Gamma \rrbracket$  — that is, a morphism  $\llbracket A \rrbracket \rightarrow \llbracket \Gamma \rrbracket$  whose fibres are the instances of  $A$  — and a term as a section of that morphism.

**Theorem 6.1** (Seely). *Let  $\mathcal{E}$  be locally cartesian closed. Substitution along  $f: \Delta \rightarrow \Gamma$  is interpreted by pullback  $f^*: \mathcal{E}/\Gamma \rightarrow \mathcal{E}/\Delta$ ; the dependent sum and product along  $f$  are interpreted by the left and right adjoints*

$$\Sigma_f \dashv f^* \dashv \Pi_f,$$

*which exist in any locally cartesian closed category; and the extensional identity type  $\text{Id}_A$  is interpreted by the diagonal  $\Delta: A \rightarrow A \times_\Gamma A$  regarded as an object of  $\mathcal{E}/(A \times_\Gamma A)$ . These assignments soundly interpret extensional Martin-Löf type theory: every derivable judgment holds in the model.*

*Proof sketch.* That  $\Sigma_f$  (composition with  $f$ ) is left adjoint to  $f^*$  is a direct verification; that  $f^*$  has a right adjoint  $\Pi_f$  is the definition of local cartesian closure. The  $\beta$ - and  $\eta$ -laws for  $\Pi$  and  $\Sigma$  are the triangle identities of these adjunctions. For the identity type, a section of  $\Delta$  pulled back along  $(a, b)$  exists iff  $a = b$  as morphisms, which is exactly the extensional reflection rule; the eliminator  $J$  is interpreted by composing with the isomorphism  $\Sigma_\Delta 1 \cong \Delta$ . Soundness is then induction on derivations.  $\square$

**Remark 6.2.** Two caveats, both well understood in the literature available to this thesis. First, pullback is functorial only up to isomorphism, while substitution in syntax is strictly functorial; the mismatch is repaired by standard strictification (split fibrations, in the sense of the fibrational semantics developed after Bénabou; see Hofmann’s construction [10]). Second, the *intensional* identity type is not interpreted by the diagonal — the reflection rule fails — and its semantics is left outside the scope of this thesis.

Since every topos is locally cartesian closed (Proposition 5.3 and [12]), every topos is a model of extensional dependent type theory. The BHK interpretation thus acquires, over any topos, the *formal* semantics promised in Chapter 4, with Kripke–Joyal forcing (Theorem 5.4) computing what the type-theoretic connectives mean stage by stage.

## 6.2 The Lambek–Scott equivalence

The passage from logic to toposes also reverses.

**Theorem 6.3** (Lambek–Scott [15]). *From every intuitionistic higher-order theory  $\mathbb{T}$  one constructs a syntactic topos  $\mathcal{E}(\mathbb{T})$  whose objects are the closed types of  $\mathbb{T}$  up to provable isomorphism and whose morphisms are the provably functional relations; and from every elementary topos  $\mathcal{E}$  one extracts its internal theory  $\text{Th}(\mathcal{E})$ . These assignments are mutually inverse up to equivalence:  $\mathcal{E}(\text{Th}(\mathcal{E})) \simeq \mathcal{E}$ , and  $\text{Th}(\mathcal{E}(\mathbb{T}))$  is a conservative extension of  $\mathbb{T}$ . Moreover  $\mathcal{E}(\mathbb{T})$  is initial among toposes containing a model of  $\mathbb{T}$ , and a sequent is derivable in  $\mathbb{T}$  iff it is valid in  $\mathcal{E}(\mathbb{T})$  — completeness, refining the soundness of Theorem 5.5.*

We do not reproduce the (long) proof; what matters here is its *shape*, identical to the shape of Theorems 3.4 and 3.7: syntax, taken up to provable equality, *is* a universe, and it is the initial one. The term algebra, the Lawvere theory, and the syntactic topos are one construction at three levels of logical strength.

## 6.3 The triangle

Assembling Theorem 3.7 (equational  $\leftrightarrow$  finite-product categories), Theorem 6.1 (dependent types  $\leftrightarrow$  locally cartesian closed categories), and Theorem 6.3 (higher-order logic  $\leftrightarrow$  toposes), and observing the inclusions

$$\text{finite products} \subset \text{local cartesian closure} \subset \text{topos structure},$$

we obtain a ladder of doctrines, each interpreting the last: every algebraic theory is a dependent theory with no dependency; every dependent theory embeds in its higher-order completion. The three foundational idioms are thereby exhibited as successive fragments of one hierarchy, with the translations given by theorems — the Unified Foundations Thesis of Chapter 1, now with content. The formal statement is deferred to Chapter 8.

## 6.4 One structure, three idioms

The abstract triangle is best seen carrying a familiar load. Consider the group concept.

*Equationally*, a group is a model of the Lawvere theory  $\mathbf{L}_{\text{Grp}}$ : a product-preserving functor  $\mathbf{L}_{\text{Grp}} \rightarrow \mathbf{Set}$  (Theorem 3.7), and this description transports to any finite-product category (Remark 3.8): a sheaf of groups is the same functor valued in  $\text{Sh}(X)$ .

*Type-theoretically*, the group concept is the dependent type

$$\text{Grp} = \sum_{G:\mathcal{U}} \sum_{m:G \times G \rightarrow G} \sum_{e:G} \sum_{i:G \rightarrow G} \text{Assoc}(m) \times \text{Unit}(m, e) \times \text{Inv}(m, e, i),$$

where the last three factors are (products of) identity types recording the axioms, per Remark 4.3. A term of this type is a group *together with* the evidence that it is one; a homomorphism is a function together with an inhabitant of the identity type asserting commutation with  $m$ . Constructions on groups become programs, and Proposition 3.11’s adjoints become computable.

*Topos-theoretically*, the group concept is a formula of the internal language: “ $m$  is associative,  $e$  neutral,  $i$  inverting” is a formula  $\gamma(m, e, i)$  of type  $\Omega$  over the object  $G^{G \times G} \times G \times G^G$ , and an internal group is a global element of its extension. Kripke–Joyal forcing (Theorem 5.4) then explains automatically what a group means *locally*: in  $\text{Sh}(X)$ , the

stalkwise groups of algebraic topology; over a Kripke poset, groups growing along stages of knowledge.

The three descriptions have different affordances — the first computes adjoints, the second computes proofs, the third computes semantics — and Theorem 8.2 will certify that they are descriptions *of the same thing*. That certification, rather than any one of the descriptions, is the foundation proposed by this thesis.

## Chapter 7

# Superposition: the Kolmogorov–Arnold Theorem

### 7.1 Statement

The thirteenth of Hilbert’s problems [9] asked whether every continuous function of several variables is a superposition of continuous functions of fewer. Kolmogorov and Arnold answered affirmatively in 1957.

**Theorem 7.1** (Kolmogorov [14], Arnold [1]). *There exist continuous monotone functions  $\psi_{pq}: [0, 1] \rightarrow \mathbb{R}$ , independent of  $f$ , such that every continuous  $f: [0, 1]^n \rightarrow \mathbb{R}$  can be written*

$$f(x_1, \dots, x_n) = \sum_{q=0}^{2n} \phi_q \left( \sum_{p=1}^n \psi_{pq}(x_p) \right)$$

for suitable continuous  $\phi_q: \mathbb{R} \rightarrow \mathbb{R}$  depending on  $f$ .

The proof is a delicate construction in real analysis and is not reproduced here; see Kolmogorov’s original note [14]. We remark only that the theorem is exact, not approximate — it is not a corollary of Stone–Weierstrass, which yields uniform approximation by polynomials but no exact representation, and the 2004 text’s appeal to Stone–Weierstrass is hereby corrected. What Stone–Weierstrass does supply is the easier, approximate analogue, and the contrast between the two is instructive: the algebraic operations (addition, multiplication) approximate everything; addition together with a fixed finite family of unary functions represents everything.

### 7.2 Superposition as clone generation

The reading this thesis places on Theorem 7.1 is algebraic. In universal algebra, the set of all operations obtainable from a given stock by composition and projection is the *clone* they generate [24, 3] — precisely the morphisms  $n \rightarrow 1$  of the Lawvere theory they present (Theorem 3.7). Hilbert’s question is then the question of clone generation for the “algebra” of continuous functions:

**Corollary 7.2.** *In the clone of continuous operations on  $[0, 1]$  (suitably extended to  $\mathbb{R}$ ), the binary operation  $+$  together with unary operations generates all continuous operations of every arity.*



Thus the analytic theorem asserts that a certain *infinitary, topological* algebraic theory is generated by its unary and binary parts — a statement of exactly the form that functorial semantics renders diagrammatic. The theorem thereby earns its place in a thesis on foundations: it exhibits the reach of the algebraic idiom (generation of all operations by composition from few) beyond the discrete settings of Chapter 3, and it does so at the boundary where algebra meets topology — the boundary the name UCAT (Chapter 8) is chosen to mark.

### 7.3 The discrete analogue

That the phenomenon is algebraic rather than analytic is confirmed by its discrete counterpart, which predates Kolmogorov’s theorem by a decade.

**Theorem 7.3** (Sierpiński [25]). *Every finitary operation on a finite set is a composition of binary operations.*

*Proof sketch.* Encode a pair of elements of the  $k$ -element set  $A$  as a single element of a  $k^2$ -element set; since one may instead work on any finite carrier by padding, an  $n$ -ary operation becomes an  $(n-1)$ -ary operation on encoded arguments, composed with the binary encoding map. Iterating reduces  $n$  to 2. On the two-element set the result sharpens: Post’s analysis [24] shows a *single* binary operation (the Sheffer stroke) generates the whole clone — the completeness of NAND familiar from Boolean logic.  $\square$

Continuity is precisely what obstructs the crude encoding argument — no continuous injection  $[0, 1]^2 \rightarrow [0, 1]$  exists, by invariance of domain — and the depth of Theorem 7.1 lies in circumventing this with addition and monotone unary maps. The pair of theorems, Sierpiński’s and Kolmogorov’s, thus mark the same foundational fact in the discrete and continuous regimes: *high arity is never fundamental; composition generates it from low arity*. Read through Theorem 3.7, this says the morphisms  $n \rightarrow 1$  of the relevant Lawvere theories are all reachable from a bounded stock — the diagrammatic form in which the present chapter hands the fact to the synthesis that follows.

## Chapter 8

# Synthesis: UCAT and the Unified Foundations Theorem

### 8.1 The unified framework

**Definition 8.1** (UCAT). *Universal Categorical Algebraic Topology is the framework in which a foundation for mathematics is specified by an elementary topos  $\mathcal{E}$  together with (i) its class of internal algebraic theories and their models in the sense of Remark 3.8; (ii) its dependent type theory in the sense of Theorem 6.1; and (iii) its internal higher-order logic in the sense of Theorems 5.4 and 6.3. A translation of foundations is a logical functor of toposes, which induces translations of all three presentations at once.*

**Theorem 8.2** (Unified Foundations Theorem). *Let  $\mathcal{E}$  be any elementary topos. Then:*

1. *every algebraic theory  $\mathbf{L}$  has a category  $\text{Mod}(\mathbf{L}, \mathcal{E})$  of internal models, and equational reasoning is sound and complete for provable equality of  $\mathbf{L}$ -morphisms (Theorems 3.4 and 3.7);*
2.  *$\mathcal{E}$  soundly interprets extensional Martin-Löf type theory, and under propositions-as-types the BHK clauses coincide with Kripke-Joyal forcing over  $\mathcal{E}$  (Theorems 6.1 and 5.4);*
3.  *$\mathcal{E}$  soundly interprets intuitionistic higher-order logic, and every such theory has an initial topos in which derivability and validity coincide (Theorems 5.5 and 6.3).*

*Consequently, any body of mathematics formalizable in equational logic, in dependent type theory, or in intuitionistic higher-order logic admits an interpretation in every elementary topos, and the three formalizations, where they overlap, are intertranslatable with preservation of provability.*

*Proof.* Each clause assembles results proved or precisely cited above. (1) Internal models are product-preserving functors  $\mathbf{L} \rightarrow \mathcal{E}$  (Remark 3.8); soundness and completeness for the equational fragment are Theorem 3.4 transported along the equivalence of Theorem 3.7, whose construction is purely syntactic and so applies verbatim with  $\mathcal{E}$  in place of **Set** on the semantic side, validity in the internal sense of Chapter 5. (2) is Theorem 6.1 applied to  $\mathcal{E}$ , which is locally cartesian closed; the identification of BHK with forcing is the content of Theorem 5.4 read clause against clause. (3) is Theorems 5.5 and 6.3. The consequence follows because the three doctrines form the ladder of Chapter 6: the inclusions of fragments induce, on any  $\mathcal{E}$ , restriction of interpretations, and the initial-topos property of Theorem 6.3 makes the translations canonical.  $\square$

*Remark 8.3.* The theorem is stated for *every* topos deliberately: foundational commitments (choice, excluded middle, infinity) reappear as *properties of the chosen  $\mathcal{E}$* , not as global legislation. In **Set** one recovers classical mathematics; in sheaves, mathematics varying continuously over a space (Example 5.2); in the free topos of Theorem 6.3, exactly what is provable and no more.

## 8.2 Why “Universal Categorical Algebraic Topology”

Each word of the name earns its keep. *Universal* and *algebraic* record the first pillar and its portability across universes; *categorical* records the medium in which all three presentations live and the Yoneda-completeness of relational description (Corollary 2.4); *topology* records both the topos-theoretic semantics — whose paradigm case, sheaves on a space, is topological (Example 5.2) — and the analytic boundary marked by Chapter 7.

## 8.3 Comparison with the set-theoretic foundation

It remains to say how UCAT stands to the incumbent. Three contrasts are decisive.

*Encoding versus interface.* In ZF, the number 3, the ordered pair, and the function each receive a particular set-theoretic construction, and questions about the encoding ( $2 \in 3$ ?) are grammatical even when mathematically meaningless. In UCAT an object is specified by its universal property and determined, by Corollary 2.4, exactly up to isomorphism — the only invariance mathematics ever uses. The framework cannot even formulate the spurious questions, which is the correct amount of expressiveness.

*One universe versus a category of universes.* ZF legislates a single cumulative hierarchy. UCAT treats universes — toposes — as objects of study, with logical functors between them (Definition 8.1); passing from **Set** to sheaves or to the free topos is a change of base, not a change of subject. Independence phenomena become geometry: a statement independent of a theory is one holding in some of its toposes and failing in others, with Cohen forcing itself appearing as a sheaf construction (Chapter 5). Set theory is not thereby discarded but *located*: **Set** is the special — Boolean, choice-complete, two-valued — object of the general theory, and an elementary axiomatization of it in wholly categorical terms was given by Lawvere in 1964 [17].

*Foundation versus organization.* The set-theoretic tradition asks for a bedrock from which all else is built up; the present framework asks for a load-bearing frame through which the parts of mathematics communicate. On the second reading the demand that a foundation be *unique* lapses — what must be unique is the system of translations, and Chapters 3–6 supply exactly that.

## 8.4 Scope, limitations, and outlook

Honesty requires three concessions. First, UCAT is a *relative* foundation: it organizes and intertranslates formal systems, but by Gödel’s incompleteness theorems [7] no consistent system of the strength considered here proves its own consistency, and UCAT does not pretend otherwise. The unification claimed is of *idioms*, not a completeness claim about mathematical truth. Second, the type-theoretic pillar has been treated in its extensional form; the semantics of intensional identity types (Remark 6.2) is deliberately left open, and the author regards the question of what mathematical structure the intensional identity type describes as the most interesting one this thesis does not answer. Third, the classical

metatheory used informally here (e.g. in citing the HSP theorem) has not itself been internalized; doing so uniformly is routine but lengthy.

The outlook, as of 2004, is twofold. Mechanization: proof assistants built on dependent type theory (the tradition of Automath and its successors, e.g. Coq [4]) make the constructive pillar executable, and Theorem 8.2(2) says their reach is every topos. And geometry: the algebraic-geometric provenance of toposes suggests that the algebraic and topological readings of Chapter 7 are two ends of a single spectrum whose systematic exploration — functorial semantics valued in progressively richer geometric universes — the author intends to pursue.

## 8.5 Conclusion

The thesis set out to replace a choice among foundations with a theorem about their agreement. The materials were classical: Birkhoff’s completeness and variety theorems for the equational idiom; Martin-Löf’s calculus, realizing the BHK reading of the connectives, for the constructive idiom; the Lawvere–Tierney axioms, with the Mitchell–Bénabou language and Kripke–Joyal semantics, for the semantic idiom. The joints were equally classical: functorial semantics (Theorem 3.7), the interpretation of dependent types in locally cartesian closed categories (Theorem 6.1), and the Lambek–Scott equivalence (Theorem 6.3). What the thesis claims as its own is the assembly: the recognition that these results compose into a single statement — Theorem 8.2 — in which each idiom interprets in every topos and the translations between idioms preserve provability; the reading of the Kolmogorov–Arnold and Sierpiński theorems as one fact about the generation of operations, locating analysis inside the same architecture; and the packaging of the whole as UCAT, a foundation whose universes are objects of study rather than articles of faith.

If the perspective is right, the traditional question “what are mathematical objects?” gives way to the question “how do mathematical descriptions translate?” — and that question, unlike the first, has theorems for answers. The recurring construction of the thesis — the term algebra, the free model, the syntactic topos, each initial among its semantics — suggests that syntax itself is best understood as the initial object of the category of its interpretations. Making that suggestion precise across the full ladder of doctrines is, in the author’s estimation, the natural continuation of the work recorded here.

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